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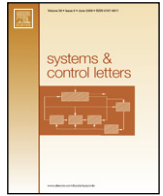
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# Quantization effects on synchronized motion of teams of mobile agents with second-order dynamics

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## ABSTRACT

For a team of mobile agents governed by second-order dynamics, this paper studies how different quantizers affect the performances of consensus-type schemes to achieve synchronized collective motion. It is shown that when different types of quantizers are used for the exchange of relative position and velocity information between neighboring agents, different collective behaviors appear. Under the chosen logarithmic quantizers and with symmetric neighbor relationships, we prove that the agents' velocities and positions get synchronized asymptotically. We show that under the chosen symmetric uniform quantizers and with symmetric neighbor relationships, the agents' velocities converge to the same value asymptotically while the differences of their positions converge to a bounded set. We also show that when the uniform quantizers are not symmetric, the agents' velocities may grow unboundedly. Through simulations we present richer undesirable system behaviors when different logarithmic and uniform quantizers are used. Such different quantization effects underscore the necessity for a careful selection of quantization strategies, especially for multi-agent systems with higher-order agent dynamics.

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## 1. Introduction

Recently significant research efforts have been made to study how to coordinate the motion of teams of mobile autonomous agents [1]. One popular approach is to use consensus-type algorithms to guide a team of agents to coincide with one another moving with the same velocity under the conditions that the relative position and/or relative velocity information is shared locally among agents and no agent is isolated from the rest of the team [2–4]. Since agents might be constrained by their limited sensing capabilities, they sometimes cannot acquire their neighboring agents' information through realtime sensing, but rely on digital communication to obtain the needed information in its quantized form. This has motivated a growing number of research activities studying how to design effective coordination control strategies using quantized information [5–12].

Agents governed by second-order dynamics as double-integrators are widely used for modeling mobile autonomous agents especially when the research focus is on the collective team dynamics instead of detailed individual agent dynamics [13]. Multi-agent systems with second-order agent dynamics can have

dramatically different collective behavior than those with first-order agent dynamics even when agents are coupled together in similar manners [14]. However, while various quantized consensus schemes have been proposed for multi-agent systems with first-order dynamics [7,10], less is known about the quantization effects on the consensus-type algorithms for motion coordination in systems with higher-order dynamics. Recently some interesting sufficient and/or necessary conditions have been constructed for synchronizing coupled double integrators without quantization [13,14]. In a more recent paper [15], higher-order passive nonlinear systems under quantized measurements are considered, but the coordination task considered there is different and its results cannot be applied directly to the problem considered here.

In this paper, we utilize the control laws that have been used in [13], but study their performances when quantized information is used. Then a new set of tools including new forms of Lyapunov functions are developed accordingly to deal with the challenges in analysis for the discontinuity on the right-hand side of the system equations as a result of quantization. We find in this paper that when the chosen logarithmic quantizers are used in the proposed coordination scheme and the neighbor relationships are symmetric, the agents' velocities and positions get synchronized asymptotically. When the chosen symmetric uniform quantizers are used instead, the agents' velocities converge to the same value asymptotically, while the differences of the agents' positions converge to a bounded set as time goes to infinity; in comparison, when the uniform quantizers are asymmetric, the agents' velocities might

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keep increasing and become unbounded. We also indicate through simulations that richer undesirable system behavior may appear under the chosen uniform and logarithmic quantizers, e.g. the agents' positions may never become the same. Some of such undesirable behaviors are inherently associated with the higher-order agent dynamics. Hence, it is emphasized that when choosing quantization schemes for agents with higher-order dynamics, in order to achieve desired motion coordination, appropriate quantizers have to be picked carefully.

The rest of the paper is organized as follows. In Section 2, the quantized control for motion synchronization is discussed for systems of agents governed by second-order dynamics and the uniform and logarithmic quantizers are defined. We review briefly in Section 3 the tools from nonsmooth analysis that we use. The analysis for systems with the chosen logarithmic and uniform quantizers are discussed in Sections 4 and 5 respectively. We provide some additional simulation results in Section 6 for the case when the neighbor relationships are not symmetric.

## 2. Motion coordination for agents with second-order dynamics

We consider a team of  $N > 0$  autonomous agents, each of which is governed by the following second-order dynamics

$$\begin{cases} \dot{r}_i = v_i \\ \dot{v}_i = u_i \end{cases} \quad i = 1, \dots, N, \quad (1)$$

where  $r_i, v_i \in \mathbb{R}^n$  denote the position and the velocity of agent  $i$  respectively and  $u_i$  is agent  $i$ 's control input. The goal for designing distributed control laws  $u_i$  is to synchronize the motions of the  $N$  agents in such a way that the velocities and positions of all the agents become the same asymptotically and thus they move together as a single entity. Such a motion coordination problem has been studied before [13,14], and the solution that has been proposed is to use a consensus-type scheme

$$u_i = - \sum_{j \in \mathcal{N}_1(i)} (r_i - r_j) - \sum_{j \in \mathcal{N}_2(i)} (v_i - v_j), \quad (2)$$

where  $\mathcal{N}_1(i)$  (resp.  $\mathcal{N}_2(i)$ ) denotes the set of agent  $i$ 's neighbors in the graph  $\mathbb{G}_1$  (resp.  $\mathbb{G}_2$ ) that describes the neighbor relationships in terms of whether the position (resp. velocity) information can be exchanged between a pair of agents. We use  $a_{ij}$  and  $b_{ij}$ ,  $1 \leq i, j \leq N$ , to denote the elements of the adjacency matrices [16] of  $\mathbb{G}_1$  and  $\mathbb{G}_2$  respectively; in other words,  $a_{ij}$  (resp.  $b_{ij}$ ) equals one if  $j$  is a neighbor of  $i$  in  $\mathbb{G}_1$  (resp.  $\mathbb{G}_2$ ) and zero otherwise. And we set  $a_{ii} = 0, b_{ii} = 0$  for all  $i = 1, \dots, N$ .

In the sequel, we assume that  $\mathbb{G}_1$  and  $\mathbb{G}_2$  are undirected and fixed. Note that in the context of distributed control, each agent only knows the *relative* position or velocity information, i.e. no global coordinate system is available. It has been shown in [13] that when  $\mathbb{G}_1$  and  $\mathbb{G}_2$  are connected, the control law (2) can achieve the goal effectively.

In this paper, we consider the scenario where for each agent, the relative position and velocity information of its neighbors is acquired through digital communication. Hence, if we continue to use the consensus-type coordination strategy (2), we have the control signals in the following form

$$u_i = - \sum_{j \in \mathcal{N}_1(i)} \mathbf{q}(r_i - r_j) - \sum_{j \in \mathcal{N}_2(i)} \mathbf{q}(v_i - v_j), \quad (3)$$

where  $\mathbf{q} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  denotes the vector quantizer of choice. Here we have assumed that all the agents have been installed with identical quantizers.

**Remark 1.** In the literature, when quantizers are applied to agents with *first-order* dynamics, different information has been quantized. For example, in [6] the quantization takes place after the relative positions have been summed up, namely

$$u_i = -\mathbf{q}\left(\sum_{j \in \mathcal{N}_1(i)} (r_i - r_j)\right);$$

in [10] the *absolute* position information in some global coordinate system is quantized, namely

$$u_i = - \sum_{j \in \mathcal{N}_1(i)} (\mathbf{q}(r_i) - \mathbf{q}(r_j)).$$

In [17], the relative position information is quantized in a similar way for what we have done in (3) for second-order agent dynamics. But the coordination task is different, and thus the control goal is different.

In this paper, we consider the following three types of quantizers. The *symmetric uniform quantizer* we consider is a map  $q_u : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$q_u(x) = \delta_u \left( \left\lfloor \frac{x}{\delta_u} \right\rfloor + \frac{1}{2} \right), \quad (4)$$

where  $\delta_u$  is a positive number and  $\lfloor a \rfloor, a \in \mathbb{R}$ , denotes the greatest integer that is less than or equal to  $a$ . The uniform quantizer (4) is similar to those used in [8,17].

The *asymmetric uniform quantizer* we consider [18] is a map  $q_u^* : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$q_u^*(x) = \delta_u \left\lfloor \frac{x}{\delta_u} \right\rfloor. \quad (5)$$

The *logarithmic quantizer* we use [8] is an odd map  $q_l : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$q_l(x) = \begin{cases} e^{q_u(\ln x)} & \text{when } x > 0; \\ 0 & \text{when } x = 0; \\ -e^{q_u(\ln(-x))} & \text{when } x < 0. \end{cases} \quad (6)$$

Note that for the uniform quantizers, the quantization error is always bounded by  $\delta_u$ , namely  $|q_u(x) - x| \leq \delta_u$  or  $|q_u^*(x) - x| \leq \delta_u$  for all  $x \in \mathbb{R}$ . Note also that for the logarithmic quantizer, it holds that

$$x q_l(x) \geq 0, \quad \text{for all } x \in \mathbb{R}, \quad (7)$$

and the equality sign holds if and only if  $x = 0$ ; the quantization error for the logarithmic quantizer is bounded by  $|q_l(x) - x| \leq \delta_l |x|$ , where the parameter  $\delta_l$  is determined by  $\delta_l = 1 - e^{-\delta_u}$ .

The above definitions of scalar-valued uniform and logarithmic quantizers can be easily generalized to their counterparts of vector-valued quantizers. Take the logarithmic quantizer as an example. For any  $x = [x_1 \dots x_n]^T \in \mathbb{R}^n$ , we define the vector logarithmic quantizer  $\mathbf{q}_l(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  to be  $\mathbf{q}_l(x) \triangleq [q_l(x_1) \dots q_l(x_n)]^T$ . One can easily check that  $\langle x, \mathbf{q}_l(x) \rangle \geq 0$  and the equality sign holds if and only if  $x = 0$ .

The main result of the paper is to show different quantization effects on the performances of the consensus-type coordination algorithms (3). Because of the discontinuity of the quantized signals, we will make use of nonsmooth analysis of differential equations to solve our problem. We give some preliminaries on nonsmooth analysis in the next section.

## 3. Preliminaries on nonsmooth analysis

For a differential equation

$$\dot{x}(t) = X(x(t)) \quad (8)$$

where  $X : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is measurable but discontinuous, the existence of a continuously differentiable solution is not guaranteed. In this paper, we adopt the Filippov solution [19].

**Definition 1.** Let  $\mathfrak{B}(\mathbb{R}^d)$  denote the collection of all subsets of  $\mathbb{R}^d$ . The Filippov set-valued map  $F[X] : \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$  is defined by

$$F[X](x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \overline{\text{co}}\{X(\mathbf{B}(x, \delta) \setminus \mathcal{S})\}, \quad x \in \mathbb{R}^d$$

where  $\mathcal{S}$  is the set of  $x$  at which  $X(x)$  is discontinuous,  $\mathbf{B}(x, \delta)$  is the open ball of radius  $\delta$  centered at  $x$ ,  $\overline{\text{co}}$  denotes the convex closure, and  $\bigcap_{\mu(\mathcal{S})=0}$  denotes the intersection over all sets  $\mathcal{S}$  of Lebesgue measure zero.

Filippov solutions are then defined to be those absolutely continuous curves, which satisfy the differential inclusion of the form  $\dot{x}(t) \in F[X](x)$ .

The Filippov set-valued map obeys the following rule.

**Lemma 1 ([19]).** If  $X_1, X_2 : \mathbb{R}^d \rightarrow \mathbb{R}^m$  are locally bounded at  $x_0 \in \mathbb{R}^d$ , then

- (a) *sum rule:*  $F[X_1 + X_2](x_0) \subseteq F[X_1](x_0) + F[X_2](x_0)$ ,
- (b) *product rule:*  $F[(X_1, X_2)^T](x_0) \subseteq F[X_1](x_0) \times F[X_2](x_0)$ ,

for which the equality signs hold when either  $X_1$  or  $X_2$  is continuous at  $x_0$ .

A sufficient condition for the existence of the Filippov solution is given as follows.

**Lemma 2 ([19]).** Assume  $X : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is measurable and locally essentially bounded, i.e. bounded in any bounded neighborhood of every point of definition excluding the sets of measure zero. Then for all  $x_0 \in \mathbb{R}^d$ , there exists a Filippov solution to (8) with the initial condition  $x(0) = x_0$ .

We also use the following notions of the generalized directional derivative and generalized gradient.

**Definition 2 ([20]).** Assume  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is locally Lipschitz near any given point  $x \in \mathbb{R}^d$ . Then the generalized directional derivative of  $f$  at  $x$  in the direction  $v \in \mathbb{R}^d$  is defined by

$$f^0(x; v) \triangleq \limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t},$$

where  $y$  is a vector in  $\mathbb{R}^d$  and  $t$  is a positive number.

**Definition 3 ([20]).** If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is locally Lipschitz, its generalized gradient is defined by

$$\partial f(x) \triangleq \text{co}\{\lim_{i \rightarrow \infty} \nabla f(x_i) : x_i \rightarrow x, \quad f'(x_i) \text{ exists}\},$$

where  $\text{co}$  denotes the convex hull, and  $f'(x_i)$  is the derivative of  $f$  at  $x_i \in \mathbb{R}^d$ .

The relationship between the generalized directional derivative and generalized gradient can be summarized as follows.

**Lemma 3. [ [20]]** Assume  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is locally Lipschitz near  $x$ . Then for every direction  $v \in \mathbb{R}^d$ , we have

$$f^0(x; v) = \max\{\langle \zeta, v \rangle : \zeta \in \partial f(x)\},$$

where  $\langle \cdot \rangle$  denotes the inner product.

The definition of regular functions is based on the notion of right directional derivative  $f'(x; v) = \lim_{t \downarrow 0} \frac{1}{t} (f(x + tv) - f(x))$ .

**Definition 4 ([20]).** A function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is said to be regular at  $x \in \mathbb{R}^d$  if for all  $v \in \mathbb{R}^d$ , the right directional derivative  $f'(x; v)$  exists and  $f'(x; v) = f^0(x; v)$ .

We say that function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is a regular function, if it is regular everywhere in its domain.

There are sufficient conditions for a function to be regular.

**Lemma 4 ([20]).** (i) If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is continuously differentiable at  $x$ , then  $f$  is regular at  $x$ . (ii) If  $\{f_i : \mathbb{R}^d \rightarrow \mathbb{R}\}$ ,  $i = 1, \dots, m$ , is a finite family of regular functions, each of which is regular at  $x$ , then for any nonnegative scalars  $\lambda_i$ ,  $\sum_{i=1}^m \lambda_i f_i(x)$  is regular at  $x$ .

The following chain rule is useful for the calculations later on.

**Lemma 5 ([21]).** Let  $x(\cdot)$  be a Filippov solution to  $\dot{x} = X(x)$  on an interval containing  $t$ , and  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  be a Lipschitz and regular function. Then  $V(x(t))$  is absolutely continuous and  $\frac{d}{dt} V(x(t))$  exists almost everywhere

$$\frac{d}{dt} V(x(t)) \in \dot{V}(x), \quad \text{for a.e. } t \geq 0,$$

where

$$\dot{V}(x) = \bigcap_{\zeta \in \partial V(x(t))} \zeta^T F[X](x).$$

## 4. Main results

Because of the discontinuity of the quantized signals, we consider Filippov solutions  $(r, v)$  to the Eqs. (1) and (3), where the notations  $r \triangleq [r_1^T \dots r_N^T]^T$  and  $v \triangleq [v_1^T \dots v_N^T]^T$ . In other words, we consider absolutely continuous functions  $(r, v)$  such that

$$\dot{v}_i \in F[u_i] \subseteq - \sum_{j \in \mathcal{N}_1(i)} F[\mathbf{q}(r_i - r_j)] - \sum_{j \in \mathcal{N}_2(i)} F[\mathbf{q}(v_i - v_j)], \quad (9)$$

where we have used Lemma 1 to deduce the relationship between the sets.

Now we take another look at the set-valued map  $F[\cdot]$  in Eq. (9). For all  $x_0 \in \mathbb{R}^n$ , let  $\mathbf{q}(x_0^-) \triangleq \lim_{x \uparrow x_0} \mathbf{q}(x)$  and  $\mathbf{q}(x_0^+) \triangleq \lim_{x \downarrow x_0} \mathbf{q}(x)$ . We use the notation

$$\tilde{r}_{ij} \triangleq r_i - r_j$$

for  $i \neq j$ . If  $\mathbf{q}(\cdot)$  is continuous at  $\tilde{r}_{ij}$ , then it follows that  $F[\mathbf{q}(\tilde{r}_{ij})] = \mathbf{q}(\tilde{r}_{ij})$ . If on the other hand  $\mathbf{q}(\cdot)$  is discontinuous at  $\tilde{r}_{ij}$ , then  $F[\mathbf{q}(\tilde{r}_{ij})] = [\mathbf{q}(\tilde{r}_{ij}^-), \mathbf{q}(\tilde{r}_{ij}^+)]$  where for  $x = [x_1 \dots x_n]^T \in \mathbb{R}^n$ ,  $[\mathbf{q}(x^-), \mathbf{q}(x^+)]$  is defined to be  $[q(x_1^-), q(x_1^+)] \times [q(x_2^-), q(x_2^+)] \times \dots \times [q(x_n^-), q(x_n^+)]$ .

The main result of the paper is to show different quantization effects on the performances of the consensus-type coordination algorithms (3). We first study logarithmic quantizers.

### 4.1. Desired synchronized motion with logarithmic quantization

When the logarithmic quantizer is used, one can show that the distributed control law that we are using can still cause the motions of all the agents to get synchronized.

**Theorem 1.** Assume the graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$  are connected and the logarithmic quantizers  $\mathbf{q}_l(\cdot)$  are used in the control (3). Then, any Filippov solution  $(r, v)$  to the system (1) and (3), is such that the positions of all the agents converge asymptotically to  $\frac{1}{N} \sum_{j=1}^N r_j(0) + \frac{t}{N} \sum_{j=1}^N v_j(0)$ , and the velocities of all the agents converge asymptotically to  $\frac{1}{N} \sum_{j=1}^N v_j(0)$ .

To prove this theorem, we first need to prove a few facts. Since  $\mathbf{q}_l(\cdot)$  is monotonic, it is integrable. So we can define the potential energy function  $W(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $\tilde{r}_{ij}$

$$W(\tilde{r}_{ij}) = \int_0^{\tilde{r}_{ij}} \mathbf{q}_l(x) dx, \quad (10)$$

where  $W(\tilde{r}_{ij})$  is the line integral from 0 to  $\tilde{r}_{ij}$ . It is easy to check that  $W(\tilde{r}_{ij}) \geq 0$  and the equality sign holds if and only if  $\tilde{r}_{ij} = 0$ . Let  $\mathcal{S}$  denote the set of all discontinuous points of  $\mathbf{q}_l$ , and then for any  $z \in \mathcal{S}$ ,  $\lim_{\tilde{r}_{ij} \uparrow z} W'(\tilde{r}_{ij}) \neq \lim_{\tilde{r}_{ij} \downarrow z} W'(\tilde{r}_{ij})$ . So  $W(\tilde{r}_{ij})$  is not differentiable with respect to  $\tilde{r}_{ij}$  at any point in  $\mathcal{S}$ . Using the generalized gradient defined in Definition 3, one has

$$\frac{\partial W(\tilde{r}_{ij})}{\partial \tilde{r}_{ij}} = \begin{cases} \mathbf{q}_l(\tilde{r}_{ij}) & \tilde{r}_{ij} \in \mathbb{R}^n \setminus \mathcal{S}, \\ \{Q^* : Q^* \in [\mathbf{q}_l(\tilde{r}_{ij}^-), \mathbf{q}_l(\tilde{r}_{ij}^+)]\} & \tilde{r}_{ij} \in \mathcal{S}. \end{cases} \quad (11)$$

We define the kinetic energy function for the velocity  $v_i$  to be

$$U(v_i) = \frac{1}{2} v_i^T v_i. \quad (12)$$

We first prove the following result.

**Lemma 6.**  $W(\cdot)$  is regular everywhere.

**Proof.** For  $z = [z_1 \ \dots \ z_n]^T \in \mathbb{R}^n$ , it suffices to prove that for all  $v = [v_1 \ \dots \ v_n]^T \in \mathbb{R}^n$ ,  $W'(z; v) = W^0(z; v)$ . Since this holds trivially for  $z \notin \mathcal{S}$ , we only need to consider the case when  $z \in \mathcal{S}$ .

From the definition of  $W(\cdot)$ , one has

$$W(z) = \int_0^z \mathbf{q}_l(x) dx = \sum_{k=1}^n \int_0^{z_k} q_l(x) dx.$$

From Lemma 3 it follows that

$$\begin{aligned} W^0(z; v) &= \max \{ \langle \zeta, v \rangle : \zeta \in \partial W(z) \} \\ &= \max \left\{ \sum_{k=1}^n \zeta_k v_k : \zeta_k \in \frac{\partial W}{\partial z_k} \right\} \\ &= \sum_{k=1}^n \max \left\{ \zeta_k v_k : \zeta_k \in \frac{\partial W}{\partial z_k} \right\}, \end{aligned} \quad (13)$$

and the last equality follows from the fact that the sets, which  $\zeta$  and  $v$  take their values from, are rectangular. Since  $q_l(z_k^-) < q_l(z_k^+)$ , for each  $k$  one has

$$\max \left\{ \zeta_k v_k : \zeta_k \in \frac{\partial W}{\partial z_k} \right\} = q_l(z_k^+) v_k \quad (14)$$

when  $v_k > 0$  and

$$\max \left\{ \zeta_k v_k : \zeta_k \in \frac{\partial W}{\partial z_k} \right\} = q_l(z_k^-) v_k \quad (15)$$

when  $v_k < 0$ . On the other hand, the directional derivative of  $W(z)$  is

$$\begin{aligned} W'(z; v) &= \lim_{t \downarrow 0} \frac{W(z + tv) - W(z)}{t} \\ &= \lim_{t \downarrow 0} \frac{\int_z^{z+tv} \mathbf{q}_l(x) dx}{t} \\ &= \lim_{t \downarrow 0} \frac{1}{t} \sum_{k=1}^n \int_{z_k}^{z_k + tv_k} q_l(x) dx. \end{aligned} \quad (16)$$

Since

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{z_k}^{z_k + tv_k} q_l(x) dx = q_l(z_k^+) v_k \quad (17)$$

when  $v_k > 0$  and

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{z_k}^{z_k + tv_k} q_l(x) dx = q_l(z_k^-) v_k, \quad (18)$$

when  $v_k < 0$ , we know

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{z_k}^{z_k + tv_k} q_l(x) dx = \max \left\{ \zeta_k v_k : \zeta_k \in \frac{\partial W}{\partial z_k} \right\},$$

for all  $k = \{1, \dots, n\}$ .

Combining (13), (16) and (19), we arrive at  $W'(z; v) = W^0(z; v)$  for all  $v$ .  $\square$

Note that  $F[\mathbf{q}_l(r_i - r_j)]$  represents the set that is given by the interval  $[\mathbf{q}_l(\tilde{r}_{ij}^-), \mathbf{q}_l(\tilde{r}_{ij}^+)]$ . We now prove a property of the set-valued map.

**Lemma 7.**  $r_i^T F[\mathbf{q}_l(r_i - r_j)] = -r_j^T F[\mathbf{q}_l(r_j - r_i)]$ , for all  $i \neq j$ .

**Proof.** Since  $\mathbf{q}_l(\cdot)$  is an odd function, one has  $\mathbf{q}_l(r_i - r_j) + \mathbf{q}_l(r_j - r_i) = \mathbf{0}$ . Then we have  $r_i^T F[\mathbf{q}_l(r_i - r_j)] = r_i^T F[-\mathbf{q}_l(r_j - r_i)] = -r_i^T F[\mathbf{q}_l(r_j - r_i)]$ .  $\square$

We can further derive some relationships between the positions and velocities of the agents.

**Lemma 8.** It holds that

$$\begin{aligned} &\sum_{i=1}^N \sum_{j \in \mathcal{N}_1(i)} r_i^T F[\mathbf{q}_l(r_i - r_j)] \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j \in \mathcal{N}_1(i)} (r_i - r_j)^T F[\mathbf{q}_l(r_i - r_j)] \end{aligned} \quad (20)$$

and

$$\begin{aligned} &\sum_{i=1}^N \sum_{j \in \mathcal{N}_2(i)} v_i^T F[\mathbf{q}_l(v_i - v_j)] \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j \in \mathcal{N}_2(i)} (v_i - v_j)^T F[\mathbf{q}_l(v_i - v_j)]. \end{aligned} \quad (21)$$

**Proof.** We only prove (21), and (20) can be proved in a similar manner.

It suffices to prove that  $\sum_{i=1}^N \sum_{j=1}^N b_{ij} v_i^T F[\mathbf{q}_l(v_i - v_j)] = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N b_{ij} (v_i - v_j)^T F[\mathbf{q}_l(v_i - v_j)]$ . Then

$$\begin{aligned} &\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N b_{ij} (v_i - v_j)^T F[\mathbf{q}_l(v_i - v_j)] \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N b_{ij} v_i^T F[\mathbf{q}_l(v_i - v_j)] \\ &\quad - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N b_{ij} v_j^T F[\mathbf{q}_l(v_i - v_j)] \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N b_{ij} v_i^T F[\mathbf{q}_l(v_i - v_j)] \\ &\quad - \frac{1}{2} \sum_{j=1}^N \sum_{i=1}^N b_{ji} v_i^T F[\mathbf{q}_l(v_j - v_i)] \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N b_{ij} v_i^T F[\mathbf{q}_l(v_i - v_j)] \\ &\quad + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N b_{ij} v_i^T F[\mathbf{q}_l(v_i - v_j)] \\ &= \sum_{i=1}^N \sum_{j=1}^N b_{ij} v_i^T F[\mathbf{q}_l(v_i - v_j)], \end{aligned}$$



where we have used the fact that  $b_{ij} = b_{ji}$  for undirected graph  $\mathbb{G}_2$  and Lemma 7.  $\square$

The following result can be proved in a similar manner.

**Lemma 9.** *It holds that*

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=1}^N a_{ij} v_i^T F[\mathbf{q}_l(r_i - r_j)] \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} (v_i - v_j)^T F[\mathbf{q}_l(r_i - r_j)]. \end{aligned} \quad (22)$$

In order to prove the convergence result in Theorem 1, we rewrite the system dynamics (1) and (3) into

$$\begin{cases} \dot{\tilde{r}}_{ij} = v_i - v_j, & j \neq i \\ \dot{v}_i = - \sum_{j=1}^N a_{ij} \mathbf{q}_l(\tilde{r}_{ij}) - \sum_{j=1}^N b_{ij} \mathbf{q}_l(v_i - v_j), \end{cases} \quad (23)$$

using the new set of states

$$\tilde{r}_{12}, \tilde{r}_{13}, \dots, \tilde{r}_{1N}, \dots, \tilde{r}_{N1}, \tilde{r}_{N2}, \dots, \tilde{r}_{N,N-1}, v_1, \dots, v_N.$$

Then in what follows, we will carry out our analysis on solutions to (23) and in fact we will prove the convergence of  $\tilde{r}_{ij}(t)$  and  $v_i(t) - v_j(t)$ .

**Proof of Theorem 1.** Consider the following candidate Lyapunov function that is defined using the potential and kinetic energy functions defined in (10) and (12) respectively

$$V(\tilde{r}, v) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} W(\tilde{r}_{ij}) + \sum_{i=1}^N U(v_i), \quad (24)$$

where  $\tilde{r} = [\tilde{r}_{12}^T, \dots, \tilde{r}_{1N}^T, \dots, \tilde{r}_{N,N-1}^T]^T$  and  $v = [v_1^T, \dots, v_N^T]^T$ . From Lemma 6 we know that  $W(\tilde{r}_{ij})$  is regular. Then in view of Lemma 4, it follows that  $\sum_{i=1}^N \sum_{j=1}^N W(\tilde{r}_{ij})$  is also regular. Furthermore,  $U(v_i)$  is continuously differentiable, so  $V(\tilde{r}, v)$  is regular. In addition,

$$\frac{\partial V(\tilde{r}, v)}{\partial \tilde{r}_{ij}} = \frac{\partial W(\tilde{r}_{ij})}{\partial \tilde{r}_{ij}}, \quad (25)$$

where  $\frac{\partial W(\tilde{r}_{ij})}{\partial \tilde{r}_{ij}}$  are given in (11), and

$$\frac{\partial V(\tilde{r}, v)}{\partial v_i} = v_i. \quad (26)$$

Then it follows that

$$\begin{aligned} \frac{d}{dt} V(\tilde{r}, v) &= \left[ \left( \frac{\partial V(\tilde{r}, v)}{\partial \tilde{r}_{12}} \right)^T, \left( \frac{\partial V(\tilde{r}, v)}{\partial \tilde{r}_{13}} \right)^T, \dots, \right. \\ &\quad \left. \left( \frac{\partial V(\tilde{r}, v)}{\partial \tilde{r}_{N-1,N}} \right)^T, v_1^T, v_2^T, \dots, v_N^T \right]^T. \end{aligned} \quad (27)$$

Applying Lemma 5, one has

$$\frac{d}{dt} V(\tilde{r}, v) \in \dot{V}(\tilde{r}, v), \quad \text{for a.e. } t \geq 0,$$

which can be further computed by

$$\dot{V}(\tilde{r}, v) = \bigcap_{\xi \in \partial V(\tilde{r}, v)} \xi^T \left[ \dot{\tilde{r}}_{12}^T, \dot{\tilde{r}}_{13}^T, \dots, \dot{\tilde{r}}_{N,N-1}^T, F^T[\dot{v}_1], \right.$$

$$\begin{aligned} & \left. F^T[\dot{v}_2], \dots, F^T[\dot{v}_N] \right]^T \\ &= \bigcap_{\xi_{ij} \in \frac{\partial V(\tilde{r}, v)}{\partial \tilde{r}_{ij}}} \left( \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \xi_{ij}^T (v_i - v_j) + \sum_{i=1}^N v_i^T F[\dot{v}_i] \right). \end{aligned}$$

Using (25) to rewrite the intersection condition and (9) to replace  $\dot{v}$ , we have

$$\begin{aligned} \dot{V}(\tilde{r}, v) &\subseteq \bigcap_{\xi_{ij} \in \frac{\partial V(\tilde{r}, v)}{\partial \tilde{r}_{ij}}} \left( \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \xi_{ij}^T (v_i - v_j) \right. \\ &\quad - \sum_{i=1}^N \sum_{j=1}^N a_{ij} v_i^T F[\mathbf{q}_l(\tilde{r}_{ij})] \\ &\quad \left. - \sum_{i=1}^N \sum_{j \in \mathcal{N}_2(i)} v_i^T F[\mathbf{q}_l(v_i - v_j)] \right). \end{aligned}$$

From Lemma 9, we can further deduce

$$\begin{aligned} \dot{V}(\tilde{r}, v) &\subseteq \bigcap_{\xi_{ij} \in \frac{\partial V(\tilde{r}, v)}{\partial \tilde{r}_{ij}}} \left( \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \xi_{ij}^T (v_i - v_j) \right. \\ &\quad - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} (v_i - v_j)^T F[\mathbf{q}_l(\tilde{r}_{ij})] \\ &\quad \left. - \sum_{i=1}^N \sum_{j \in \mathcal{N}_2(i)} v_i^T F[\mathbf{q}_l(v_i - v_j)] \right) \\ &= \bigcap_{\xi_{ij} \in \frac{\partial V(\tilde{r}, v)}{\partial \tilde{r}_{ij}}} \left( \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} (v_i - v_j)^T \xi_{ij} \right. \\ &\quad - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} (v_i - v_j)^T [\mathbf{q}_l(\tilde{r}_{ij}^-), \mathbf{q}_l(\tilde{r}_{ij}^+)] \\ &\quad - \sum_{i=1}^N \sum_{j \in \mathcal{N}_2(i)} v_i^T F[\mathbf{q}_l(v_i - v_j)] \\ &\quad = \bigcap_{\xi_{ij} \in [\mathbf{q}_l(\tilde{r}_{ij}^-), \mathbf{q}_l(\tilde{r}_{ij}^+)]} \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} (v_i - v_j)^T \\ &\quad \times [-\mathbf{q}_l(\tilde{r}_{ij}^+) + \xi_{ij}, -\mathbf{q}_l(\tilde{r}_{ij}^-) + \xi_{ij}] \\ &\quad \left. - \sum_{i=1}^N \sum_{j \in \mathcal{N}_2(i)} v_i^T F[\mathbf{q}_l(v_i - v_j)]. \right) \end{aligned}$$

Since

$$\bigcap_{\xi_{ij} \in [\mathbf{q}_l(\tilde{r}_{ij}^-), \mathbf{q}_l(\tilde{r}_{ij}^+)]} [-\mathbf{q}_l(\tilde{r}_{ij}^+) + \xi_{ij}, -\mathbf{q}_l(\tilde{r}_{ij}^-) + \xi_{ij}] = \{\mathbf{0}\},$$

and in view of Lemma 8, one has

$$\begin{aligned} \frac{d}{dt} V(\tilde{r}, v) &\in \dot{V}(\tilde{r}, v) \subseteq - \sum_{i=1}^N \sum_{j \in \mathcal{N}_2(i)} v_i^T F[\mathbf{q}_l(v_i - v_j)] \\ &= - \frac{1}{2} \sum_{i=1}^N \sum_{j \in \mathcal{N}_2(i)} (v_i - v_j)^T F[\mathbf{q}_l(v_i - v_j)]. \end{aligned} \quad (28)$$

This implies that  $\frac{d}{dt} V(\tilde{r}, v) \leq 0$ . Thus,  $V(\tilde{r}, v) \leq V(\tilde{r}(0), v(0))$ , which further implies that both  $\tilde{r}(t)$  and  $v(t)$  are bounded. Now

we apply LaSalle's invariance principle [19, Theorem 2] to show the convergence of solutions to (23). Define  $\mathcal{T} \triangleq \{(\tilde{r}, v) | V(\tilde{r}, v) \leq V(\tilde{r}(0), v(0))\}$  and  $\mathcal{E} \triangleq \{(\tilde{r}, v) | 0 \in \dot{V}(\tilde{r}, v)\}$ . Note that from (28) and the connectivity of  $\mathbb{G}_2$ ,  $\mathcal{E} = \{(\tilde{r}, v) | v_i = v_j, \forall i \neq j\}$ . The solutions to (23) converge to the largest weakly invariant set  $\mathcal{M}$  contained in  $\mathcal{T} \cap \mathcal{E}$ . Consider a solution to (23) that evolves in this set for all  $t \geq 0$ . It satisfies

$$\begin{cases} \dot{\tilde{r}}_{ij} = \mathbf{0}, & j \neq i \\ \dot{v}_i = -\sum_{j=1}^N a_{ij} \mathbf{q}_l(\tilde{r}_{ij}) & \forall i, j = 1, \dots, N. \end{cases} \quad (29)$$

Hence, the solutions to (23) converge to a set of points  $(\tilde{r}, v)$  such that every  $\tilde{r}_{ij}$  remains constant and all the velocities  $v_i$  are equal.

Following [22,15], to proceed further in the proof, we use  $D_1$  to denote the incidence matrix associated with the graph  $\mathbb{G}_1$  and introduce the variable  $z \triangleq (D_1^T \otimes I_n)r$  to denote the vector of the relative positions between neighboring agents where  $\otimes$  denotes the Kronecker product. Since  $\mathbb{G}_1$  is connected,  $\tilde{r}$  is constant if and only if  $z$  is constant. Moreover, in view of (3), the second equation in (29) can be written in a compact form

$$\dot{v} = -(D_1 \otimes I_n) \mathbf{q}_l(z).$$

Hence, a solution to (29) such that  $\tilde{r}$  is constant and  $v_i = v_j$  for all  $i, j$  in the coordinates  $(z, v)$  satisfies

$$\begin{cases} \dot{z} = \mathbf{0}, \\ \dot{v} = -(D_1 \otimes I_n) \mathbf{q}_l(z) \end{cases} \quad (30)$$

and is such that  $z = (D_1^T \otimes I_n)r$  is constant and  $(D_1^T \otimes I_n)v = \mathbf{0}$ , i.e. all the velocities are the same. Consider a solution to the system (30) and define  $\tilde{v} \triangleq (D_1^T \otimes I_n)v$ . We have  $\dot{\tilde{v}} \in -(D_1^T \otimes I_n)(D_1 \otimes I_n)F[\mathbf{q}_l(z)]$ . For a solution to (30) to remain in a set where  $z = (D_1^T \otimes I_n)r$  is constant and  $\tilde{v} = \mathbf{0}$ , it must be true that

$$\mathbf{0} \in -(D_1^T \otimes I_n)(D_1 \otimes I_n)F[\mathbf{q}_l(z)] = -(D_1^T D_1 \otimes I_n)F[\mathbf{q}_l(z)].$$

Let  $w \in F[\mathbf{q}_l(z)]$  be such that  $\mathbf{0} = (D_1^T D_1 \otimes I_n)w$ . Then  $y \triangleq (D_1 \otimes I_n)w$  belongs to  $\ker(D_1^T \otimes I_n)$ , i.e.  $(D_1^T \otimes I_n)y = \mathbf{0}$ . Then  $y^T y = y^T (D_1 \otimes I_n)w = 0$ . Hence,  $(D_1 \otimes I_n)w = \mathbf{0}$  with  $w \in F[\mathbf{q}_l(z)]$ . Since  $z = (D_1^T \otimes I_n)r$ , from  $(D_1 \otimes I_n)w = \mathbf{0}$  one obtains that  $r^T (D_1 \otimes I_n)w = 0 = z^T w$ . Since  $w \in F[\mathbf{q}_l(z)]$  and  $\mathbf{q}_l$  is the logarithmic quantizer, then  $z^T w = 0$  implies necessarily that  $z = \mathbf{0}$ . Hence a weakly invariant set for (30) where  $z = (D_1^T \otimes I_n)r$  remains constant and  $(D_1^T \otimes I_n)v = \mathbf{0}$  is such that  $z = \mathbf{0}$ . Bearing in mind the second equation of (30), this also implies that  $\dot{v} = \mathbf{0}$ . In the coordinates  $(\tilde{r}, v)$ , we conclude that the solutions converge to a set  $\mathcal{M}$  where  $\tilde{r} = \mathbf{0}$ ,  $(D_1^T \otimes I_n)v = \mathbf{0}$  and  $\dot{v} = \mathbf{0}$ .

One can further calculate the asymptotic positions and velocities for all the agents. On one hand, one can check that for any solution  $(r, v)$ ,  $(\mathbf{1}_N \otimes I_n)^T \dot{v} = \mathbf{0}$  for a.e.  $t \geq 0$ , namely  $\sum_{i=1}^N \dot{v}_i = \mathbf{0}$ . Hence, one has  $\sum_{i=1}^N v_i = \sum_{i=1}^N v_i(0)$ . Combined with the fact that  $v_i = v_j = \bar{v}$  on  $\mathcal{M}$ , we know that  $\bar{v} = \frac{1}{N} \sum_{i=1}^N v_i(0)$ , for all  $i = 1, 2, \dots, N$ . So any solution  $v_i(t)$  tends to  $\frac{1}{N} \sum_{i=1}^N v_i(0)$  as  $t \rightarrow +\infty$ . On the other hand, on  $\mathcal{M}$ ,  $\dot{r}_i = \bar{v} = \frac{1}{N} \sum_{i=1}^N v_i(0)$ , and hence  $r_i(t) = \frac{1}{N} \sum_{j=1}^N r_j(0) + \frac{t}{N} \sum_{j=1}^N v_j(0)$ , for all  $i = \{1, 2, \dots, N\}$ . We conclude that any solution  $(\tilde{r}, v)$  is such that  $\tilde{r}_i \rightarrow \frac{1}{N} \sum_{j=1}^N r_j(0) + \frac{t}{N} \sum_{j=1}^N v_j(0)$  as  $t \rightarrow +\infty$ .  $\square$

**Remark 2.** In the proof of Theorem 1, from  $0 \in \dot{V}(\tilde{r}, v)$  it is shown that  $v_i = v_j, \forall i \neq j$ , which implies that the velocities of all the agents get synchronized precisely. However, if we use a

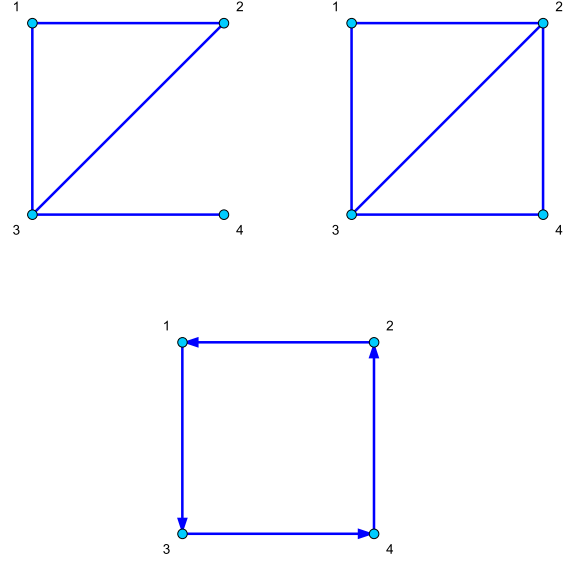


Fig. 1. Some graphs for different communication topologies.

logarithmic quantizer with finite quantization levels towards the origin, such as

$$q_l(x) = \begin{cases} e^{q_u(\ln x)} & \text{when } x > \Delta; \\ 0 & \text{when } x \in [-\Delta, \Delta]; \\ -e^{q_u(\ln(-x))} & \text{when } x < -\Delta, \end{cases} \quad (31)$$

where  $\Delta \in \mathbb{R}$  is a positive constant, the convergence result for velocities will be different. In fact, a slight modification of the proof leads to the conclusion that  $v_i - v_j \in [-\Delta, \Delta] \mathbf{1}_n, \forall j \in \mathcal{N}_2(i), i = 1, \dots, N$ ; in other words, the norms  $\|v_i - v_j\|_2$  of the relative velocities between neighboring agents are bounded from above by the constant  $\sqrt{n}\Delta$ .

Theorem 1 can be validated through simulations. We consider a team of 4 agents whose neighbor relationship graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$  are taken to be the same as shown in the upper left corner of Fig. 1.

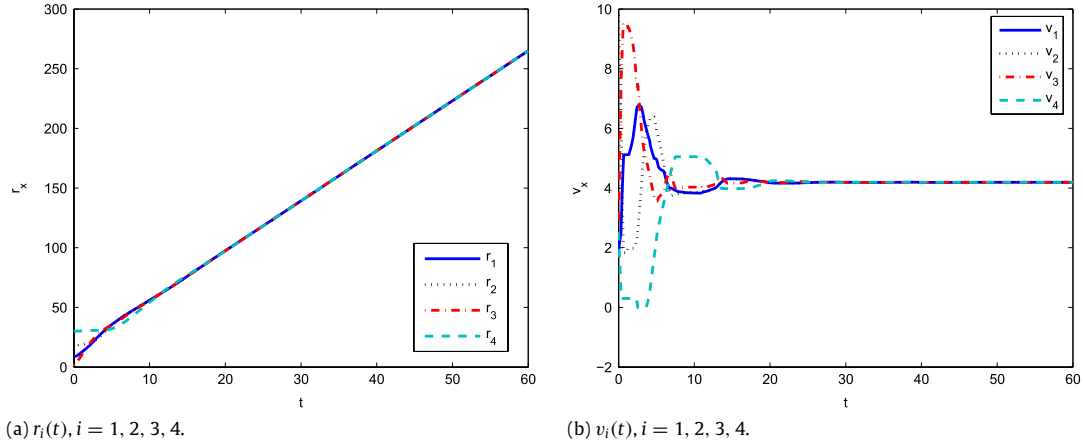
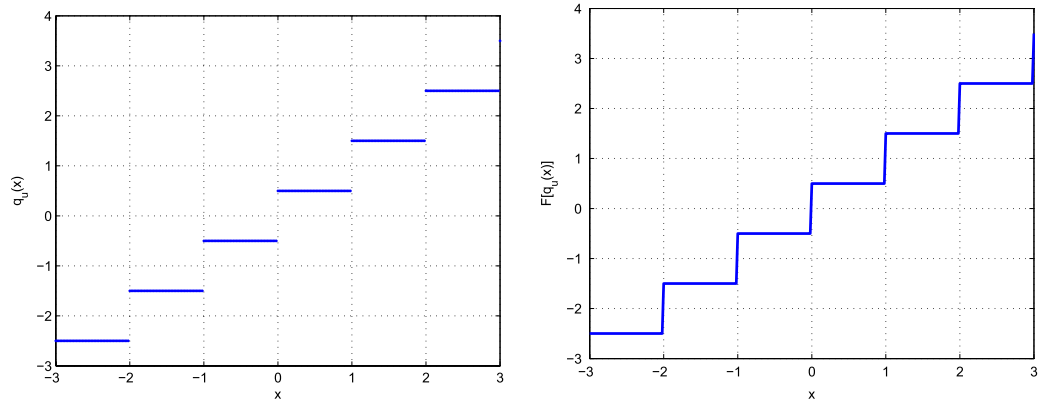
We take  $n = 2$  and let  $\delta_u = 1, \delta_l = 1 - e^{-1}$ . Each coordinate of the initial positions are chosen randomly from  $(0, 30)$  while those of the initial velocities from  $(0, 10)$ . Fig. 2 (a) and (b) illustrate the evolutions of the positions and velocities of the four agents in their  $x$ -coordinates respectively.

Furthermore, we look at the relation between the coarseness of the logarithmic quantizers and the system's convergence speed. We set three different values for  $\delta_l$  in order to compare the corresponding convergence speeds of the quantized system. And we keep all the other settings the same as in the above simulation. First, we set  $\delta_u = 1, \delta_l = 1 - e^{-1}$ . Our simulation shows that it takes 19.8 time units for the quantized system to converge. Then we set  $\delta_u = 2, \delta_l = 1 - e^{-2}$ . It takes 29.3 time units for the quantized system to converge. Finally, we set  $\delta_u = 5, \delta_l = 1 - e^{-5}$ . And it takes 131.7 time units for the quantized system to converge. The results are obtained by averaging three independent runs. The above comparison shows that the coarser the logarithmic quantizer is, the more time it takes for the quantized system to converge.

#### 4.2. Synchronized motion with uniform quantization

When uniform quantizers (4) are used, one achieves a form of practical synchronized motion.

For simplicity, we suppose that the undirected graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$  are the same and use a common symbol  $\mathbb{G}$ . Let  $L \in \mathbb{R}^{N \times N}$  denote the Laplacian matrix of the graph  $\mathbb{G}$  and  $D \in \mathbb{R}^{N \times m}$  the

Fig. 2. Evolutions of  $r_i(t)$  and  $v_i(t)$  with the logarithmic quantizers.Fig. 3. Visualization of the map  $q_u(\cdot)$  in (4) and the set-valued map  $F[q_u(\cdot)]$ , when  $\delta_u = 1$ .

corresponding incidence matrix, where  $m$  is the number of edges in the graph  $\mathbb{G}$ .

**Theorem 2.** Assume the graph  $\mathbb{G}$  is connected and that the uniform quantizers  $q_u(\cdot)$  in (4) are used in the control law (3). Then any Filippov solution  $(r, v)$  to the system (1) and (3) is such that

- (a) the velocities  $v$  of all the agents converge asymptotically to  $\frac{1}{N} \sum_{j=1}^N v_j(0)$ ;
- (b) the distance  $r_i - r_j$  between any pair of neighboring agents converge asymptotically to a set where  $\|r_i - r_j\|_2 \leq \sqrt{n} \delta_u$ ;
- (c) the positions of all the agents converge asymptotically to the set

$$\mathcal{R} = \{r \in \mathbb{R}^{Nm} : \|r - (\mathbf{1}_N \otimes I_n) r_{avg}\|_2 \leq \sqrt{\frac{mn}{\lambda_2(L)}} \delta_u\},$$

$$\text{where } r_{avg} = \frac{1}{N} \sum_{i=1}^N r_i(0) + \frac{t}{N} \sum_{i=1}^N v_i(0).$$

Here and thereafter, we use  $\|\cdot\|_2$  to denote the Euclidean norm of a vector, and  $\|\cdot\|_\infty$  to denote its  $\infty$ -norm.

To prove this theorem, we first need a few facts. The following is trivial:

**Lemma 10.** For the incidence matrix  $D \in \mathbb{R}^{N \times m}$  associated with the graph  $G$ , the null space of  $D^T D$  is the null space of  $D$ .

Moreover, we have (see e.g. Lemma 1 in [10])

**Lemma 11.** For any  $x \in \mathbb{R}^N$ , one has  $x^T L x \geq \lambda_2(L) \|x - \frac{\mathbf{1}^T x}{N} \mathbf{1}\|_2^2$ , where  $\mathbf{1} \in \mathbb{R}^N$  is the vector of all ones and  $\lambda_2$  is the algebraic connectivity.

In the following, we analyze a few properties of the uniform quantizer (4). First we represent the map  $q_u(\cdot)$  and the set-valued map  $F[q_u(\cdot)]$  in Fig. 3. This will be helpful in our analysis. From the definition of the uniform quantizer (4), one has the following lemma:

**Lemma 12.** (a) For  $x \in \mathbb{R}$  and  $|x| \leq \delta_u$ , it holds that

$$x(F[q_u(x)] + F[q_u(0)]) \subseteq [0, +\infty). \quad (32)$$

(b) For  $x \in \mathbb{R}$  and  $|x| > \delta_u$ , it holds that

$$x(F[q_u(x)] + F[q_u(0)]) \subseteq (0, +\infty). \quad (33)$$

**Proof.** We use Fig. 3 to help our analysis.

(a) If  $x = 0$ , it follows that  $x(F[q_u(x)] + F[q_u(0)]) = \{0\}$ . If  $0 < x < \delta_u$ , it follows that  $F[q_u(x)] = \{\frac{\delta_u}{2}\}$ . Then  $F[q_u(x)] + F[q_u(0)] = [0, \delta_u]$ , noting that  $F[q_u(0)] = [-\frac{\delta_u}{2}, \frac{\delta_u}{2}]$ . Thus one has that every element in the set  $x(F[q_u(x)] + F[q_u(0)])$  is nonnegative.

Similarly one proves that if  $-\delta_u < x < 0$ , then  $F[q_u(x)] + F[q_u(0)] = [-\delta_u, 0]$ . Thus one has that every element in the set  $x(F[q_u(x)] + F[q_u(0)])$  is nonnegative.

If  $x = \delta_u$  or  $x = -\delta_u$ , analogous arguments work as well. Hence, we conclude that (32) holds if  $|x| \leq \delta_u$ .

(b) If  $x > \delta_u$ , it follows that  $F[q_u(x)] \subseteq [\frac{3\delta_u}{2}, +\infty)$ . Then  $F[q_u(x)] + F[q_u(0)] \subseteq [\delta_u, +\infty)$ , noting that  $F[q_u(0)] = [-\frac{\delta_u}{2}, \frac{\delta_u}{2}]$ . Thus one has that any element in the set  $x(F[q_u(x)] + F[q_u(0)])$  is strictly positive. In the same way one can prove (33) if  $x < -\delta_u$ . Now we conclude that (33) holds if  $|x| > \delta_u$ .  $\square$



Now we are ready to prove the synchronization of the second-order systems with uniform quantizers.

**Proof of Theorem 2.** We use the variable  $z = (D^T \otimes I_n)r$  to denote the vector of the relative positions between neighboring agents. Then we rewrite (1) and (3) into the compact form

$$\begin{cases} \dot{z} = (D^T \otimes I_n)v \\ \dot{v} = -(D \otimes I_n)\mathbf{q}_u(z) - (D \otimes I_n)\mathbf{q}_u((D^T \otimes I_n)v). \end{cases} \quad (34)$$

We adopt the Lyapunov function

$$V(z, v) = \sum_{k=1}^m \sum_{j=1}^n \int_0^{z_{kj}} q_u(s) ds + \sum_{i=1}^N U(v_i)$$

where  $U(v_i) = \frac{1}{2} v_i^T v_i$ . The function is convex and when computed along the solutions to (34) it satisfies  $\frac{d}{dt} V(z(t), v(t)) \in \dot{V}(z(t), v(t))$  for a.e.  $t$ , where the set-valued derivative  $\dot{V}(z, v)$  can be computed similarly as in Lemma 1 in [15], and is given by

$$\begin{aligned} \dot{V}(z, v) &= \{a \in \mathbb{R} : \exists w \in F[\mathbf{q}_u((D^T \otimes I_n)v)] \\ &\text{s.t. } a = -v^T (D \otimes I_n)w\}. \end{aligned}$$

Let conventionally  $\max \dot{V}(z, v) = -\infty$  if  $\dot{V}(z, v) = \emptyset$ . By definition of  $F[\mathbf{q}_u((D^T \otimes I_n)v)]$  this implies that  $\frac{d}{dt} V(z, v) \subseteq (-\infty, 0]$  and that  $\mathbf{0} \in \dot{V}(z, v)$  necessarily implies  $(D^T \otimes I_n)v = \mathbf{0}$ , i.e.  $v = \text{span}\{\mathbf{1} \otimes I_n\}$ . We conclude that the solutions converge to a subset of the largest weakly invariant set where  $v = \text{span}\{\mathbf{1} \otimes I_n\}$ . On this invariant set the system evolves as

$$\begin{cases} \dot{z} = \mathbf{0} \\ \dot{v} = -(D \otimes I_n)\mathbf{q}_u(z) - (D \otimes I_n)\mathbf{q}_u((D^T \otimes I_n)v) \end{cases}$$

and the solutions must satisfy the differential inclusion

$$\begin{pmatrix} \dot{z} \\ \dot{v} \end{pmatrix} \in \begin{pmatrix} \mathbf{0} \\ -(D \otimes I_n)F[\mathbf{q}_u(z)] - (D \otimes I_n)F[\mathbf{q}_u(\mathbf{0})] \end{pmatrix}. \quad (35)$$

Consider the solution to the system (35), which evolves in the largest weakly invariant set where  $(D^T \otimes I_n)v \equiv \mathbf{0}$ . One has  $(D^T \otimes I_n)\dot{v} \equiv \mathbf{0}$ . From (35), it follows that  $(D^T \otimes I_n)\dot{v} \in -(D^T D \otimes I_n)(F[\mathbf{q}_u(z)] + F[\mathbf{q}_u(\mathbf{0})])$  and also

$$\mathbf{0} \in -(D^T D \otimes I_n)(F[\mathbf{q}_u(z)] + F[\mathbf{q}_u(\mathbf{0})]).$$

Applying Lemma 10, one has

$$\mathbf{0} \in -(D \otimes I_n)(F[\mathbf{q}_u(z)] + F[\mathbf{q}_u(\mathbf{0})]).$$

And multiplying on the right by  $r^T$ , one further obtains

$$0 \in -z^T (F[\mathbf{q}_u(z)] + F[\mathbf{q}_u(\mathbf{0})]). \quad (36)$$

The latter in combination with Lemma 12 shows that  $\|z\|_\infty \leq \delta_u$ . Thus, the solutions  $(z, v)$  converge to a set where  $\|z\|_\infty \leq \delta_u$ ,  $(D^T \otimes I_n)v = \mathbf{0}$ .

As to the further calculation of asymptotic velocities of all the agents, one can follow the argument of the last part of the proof of Theorem 1. And we have that any solution  $v_i(t)$  tends to  $\frac{1}{N} \sum_{i=1}^N v_i(0)$  as  $t \rightarrow +\infty$ . Now we calculate the asymptotic positions of all the agents. From  $\|z\|_\infty \leq \delta_u$ , one has

$$\|r_i - r_j\|_2 \leq \sqrt{n}\delta_u, \quad (37)$$

where  $i = \{1, \dots, N\}$  and  $j$  is a neighbor of  $i$ . Note that  $z = (D^T \otimes I_n)r \in \mathbb{R}^{mn}$ . Then one has

$$\|(D^T \otimes I_n)r\|_2 = \|z\|_2 \leq \sqrt{mn} \|z\|_\infty \leq \sqrt{mn}\delta_u.$$

From Lemma 11, one has

$$\begin{aligned} \|r - \left(\frac{\mathbf{1}_N \mathbf{1}_N^T}{N} \otimes I_n\right)r\|_2^2 &\leq \frac{1}{\lambda_2(L)} r^T (L \otimes I_n)r \\ &= \frac{1}{\lambda_2(L)} r^T (DD^T \otimes I_n)r \\ &= \frac{1}{\lambda_2(L)} \|(D^T \otimes I_n)r\|_2^2 \\ &\leq \frac{mn}{\lambda_2(L)} \delta_u^2, \end{aligned}$$

that is,

$$\|r - \left(\frac{\mathbf{1}_N \mathbf{1}_N^T}{N} \otimes I_n\right)r\|_2 \leq \sqrt{\frac{mn}{\lambda_2(L)}} \delta_u. \quad (38)$$

Furthermore,  $\sum_{i=1}^N \dot{r}_i(t) = \sum_{i=1}^N v_i(t) = \sum_{i=1}^N v_i(0)$ , and hence the average position of all the agents can be calculated as

$$r_{avg} \triangleq \frac{1}{N} \sum_{i=1}^N r_i(t) = \frac{1}{N} \sum_{i=1}^N r_i(0) + \frac{t}{N} \sum_{i=1}^N v_i(0). \quad (39)$$

Then in combination with (38), we have

$$\|r - (\mathbf{1}_N \otimes I_n)r_{avg}\|_2 \leq \sqrt{\frac{mn}{\lambda_2(L)}} \delta_u. \quad (40)$$

We conclude that the asymptotic positions of all the agents converge to the set  $\mathcal{R} = \{r \in \mathbb{R}^{Nn} : \|r - (\mathbf{1}_N \otimes I_n)r_{avg}\|_2 \leq \sqrt{\frac{mn}{\lambda_2(L)}} \delta_u\}$ , where  $r_{avg} = \frac{1}{N} \sum_{i=1}^N r_i(0) + \frac{t}{N} \sum_{i=1}^N v_i(0)$ .  $\square$

**Remark 3.** In the proof it is shown that  $(D^T \otimes I_n)v = \mathbf{0}$ , which means that the velocities of all the agents accurately achieve synchronization. However, if we use a different uniform quantizer, such as the one used in [7,10]:

$$q_u(x) = \delta_u \left\lfloor \frac{x}{\delta_u} + \frac{1}{2} \right\rfloor, \quad (41)$$

instead of the uniform quantizer in (4), the convergence result for velocities will be different. In fact, a slight variation of the proof shows that  $(D^T \otimes I_n)v \in [-\frac{\delta_u}{2}, \frac{\delta_u}{2}] \mathbf{1}_{mn}$ , i.e. the norm  $\|v_i - v_j\|$  of the relative velocity between neighbors is bounded by the constant  $\frac{\sqrt{n}}{2} \delta_u$ .

Theorem 2 and Remark 3 can be validated through simulations. We take the neighbor relationship graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$  both to be the graph on the upper right corner in Fig. 1. We set  $\delta_u = 1$  and initialize the system in the same way as what we have done for the simulation of the system with the logarithmic quantizer. We show the results in Fig. 4. When the uniform quantizer (4) is adopted, the agents' velocities converge<sup>1</sup> to the average value, as shown in Fig. 4(a). When the uniform quantizer (41) is adopted, the agents' velocities converge to a bound set with the diameter less than 1, shown in Fig. 4(b). The results in Fig. 4 are consistent with the different convergence results in Theorem 2 and Remark 3.

While the steady-state performances of the consensus-type coordination algorithm is satisfactory when the above quantizers are chosen, we show in the next section that this is not the case if uniform quantizers are used differently.

<sup>1</sup> Namely, every Filippov solution converges to a set where velocities synchronize. However, chattering with very small amplitudes (less than 0.05 in the shown simulation run) takes place in steady states. This is due to sliding modes along the synchronized manifolds of velocities.

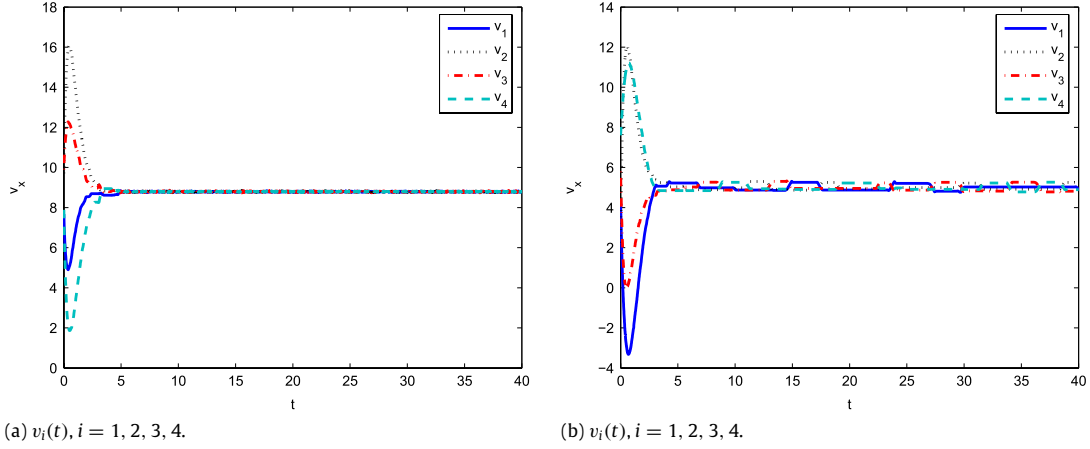


Fig. 4.  $v_i(t)$  corresponding to the uniform quantizers (4) and (41) respectively.

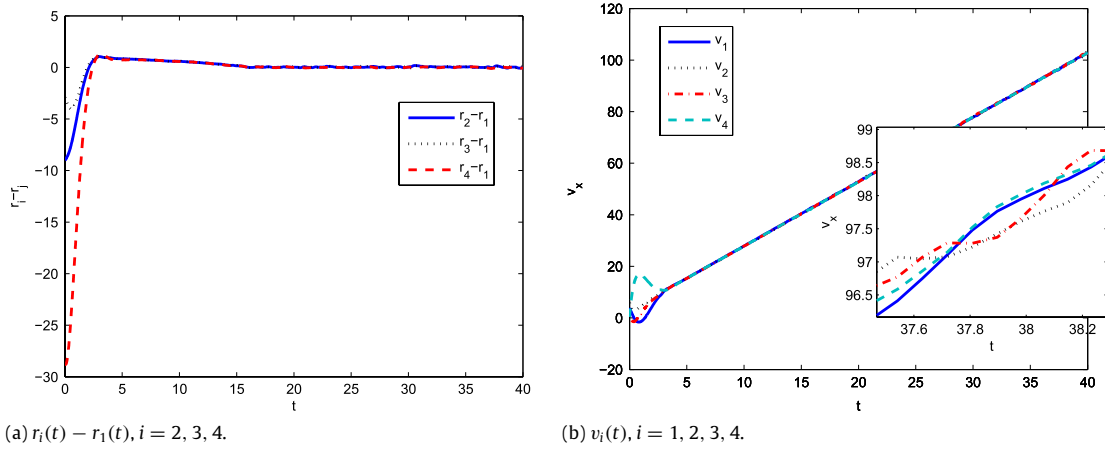


Fig. 5. Synchronized motion with unbounded agent velocities with the uniform quantizers (5).

## 5. Undesirable steady-state dynamics with asymmetric uniform quantizers

In this section, we consider effects of asymmetric uniform quantizers (5) on the consensus-type scheme (3). We first use two examples to demonstrate that when the uniform quantizers (5) are utilized for the controllers (3), some undesirable steady-state behaviors may arise for the multi-agent systems (1). In the first example, we show that, although the agents may get synchronized in the sense that they move with almost the same velocity and almost the same time-varying position asymptotically, the agents' velocities grow unboundedly, which cannot happen in reality. In the second case, we show an even worse case when the agents' positions never coincide with one another.

We take the neighbor relationship graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$  both to be the graph on the upper right corner in Fig. 1. We set  $\delta_u = 1$  and initialize the system in the same way as what we have done for the simulation of the system with the logarithmic quantizer in Section 4. We show the simulation results in Fig. 5. It is clear that as the system evolves, the agents' positions become the same, their velocities keep oscillating around the time-varying average velocity of the whole group  $\frac{1}{N} \sum_{i=1}^N v_i(t)$ . Obviously, as indicated by Fig. 5(b), the agents' velocities grow unboundedly as  $t$  increases.

Next we show that the steady states of the system can be even more undesirable, namely the agents' positions always differ from one another. Towards this end, we take the neighbor relationship graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$  both to be the graph on the upper left corner of Fig. 1. We keep all the other setting the same as before. The

simulated system dynamics are shown in Fig. 6. It is clear that the agents' positions do not become the same while their velocities oscillate around the average velocity  $\frac{1}{N} \sum_{i=1}^N v_i(t)$ . In particular, in Fig. 6(a) the values of  $r_1$ ,  $r_2$ , and  $r_3$  become the same for almost every  $t$  and  $r_4$  keeps a distance of 1 from the rest.

Now we explain the observed behavior in Fig. 5(b) and Fig. 6(b). We consider again the Filippov solutions to system (1) and (3) with the chosen uniform quantizer. To simplify the discussion, here we focus on the case when the positions  $r_i$  and the velocities  $v_i$ ,  $i = 1, \dots, N$ , are scalars. The analysis can be extended straightforwardly to the higher dimensional case. From the definition for the uniform quantizer in (5), we know that

$$\begin{aligned} F[q_u^*(r_i - r_j)] + F[q_u^*(r_j - r_i)] \\ = q_u^*(r_i - r_j) + q_u^*(r_j - r_i) = -\delta_u \end{aligned} \quad (42)$$

when  $r_i - r_j \neq k\delta_u$ , where  $k$  are integers, and

$$\begin{aligned} F[q_u^*(r_i - r_j)] + F[q_u^*(r_j - r_i)] \\ = [(k-1)\delta_u, k\delta_u] + [-(k+1)\delta_u, -k\delta_u] \\ = -\delta_u [0, 2] \end{aligned} \quad (43)$$

when  $r_i - r_j = k\delta_u$ . From (43), we have

$$\begin{aligned} \sum_{i=1}^N \sum_{j=1}^N a_{ij} F[q_u^*(r_i - r_j)] \\ = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} F[q_u^*(r_i - r_j)] \end{aligned}$$

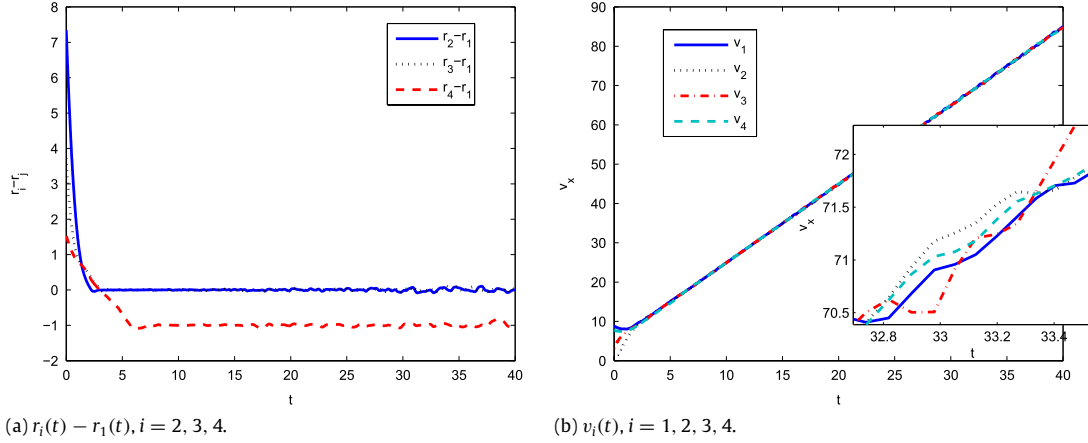


Fig. 6. Agents evolving with different positions and increasing velocities with the uniform quantizers (5).

$$\begin{aligned}
 & + \frac{1}{2} \sum_{j=1}^N \sum_{i=1}^N a_{ji} F[q_u^*(r_j - r_i)] \\
 & = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} F[q_u^*(r_i - r_j)] \\
 & + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} F[q_u^*(r_j - r_i)] \\
 & = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \{F[q_u^*(r_i - r_j)] + F[q_u^*(r_j - r_i)]\} \\
 & = -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \delta_u [0, 2].
 \end{aligned}$$

Then in combination with (9), we have

$$\begin{aligned}
 \sum_{i=1}^N \dot{v}_i & \in -\sum_{i=1}^N \sum_{j=1}^N a_{ij} F[q_u^*(r_i - r_j)] - \sum_{i=1}^N \sum_{j=1}^N b_{ij} F[q_u^*(v_i - v_j)] \\
 & = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \delta_u [0, 2] + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N b_{ij} \delta_u [0, 2]. \quad (44)
 \end{aligned}$$

Now we claim that  $\sum_{i=1}^N \dot{v}_i$  is always positive whenever there is at least one pair of  $\hat{i}$  and  $\hat{j}$ ,  $1 \leq \hat{i}, \hat{j} \leq N$ , such that  $r_{\hat{i}} - r_{\hat{j}} \neq k_1 \delta_u$  or  $v_{\hat{i}} - v_{\hat{j}} \neq k_2 \delta_u$  for any integers  $k_1$  and  $k_2$ . This is because for such a pair of  $\hat{i}$  and  $\hat{j}$ , it follows from (42) and (44) that

$$\begin{aligned}
 \sum_{i=1}^N \dot{v}_i & \in \left\{ \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} - 1 \right\} \delta_u [0, 2] \\
 & + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N b_{ij} \delta_u [0, 2] + \delta_u \quad (45)
 \end{aligned}$$

when  $r_{\hat{i}} - r_{\hat{j}} \neq k_1 \delta_u$  and

$$\begin{aligned}
 \sum_{i=1}^N \dot{v}_i & \in \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \delta_u [0, 2] \\
 & + \left\{ \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N b_{ij} - 1 \right\} \delta_u [0, 2] + \delta_u \quad (46)
 \end{aligned}$$

when  $v_{\hat{i}} - v_{\hat{j}} \neq k_2 \delta_u$ . So in either case,  $\sum_{i=1}^N \dot{v}_i$  is always positive since the first and second terms of (45) and (46) are always

nonnegative and the third terms are always positive. This gives one of the reasons that the agents' velocities may grow unboundedly as  $t$  increases.

## 6. More complicated behaviors when $\mathbb{G}_1$ and $\mathbb{G}_2$ are directed

Up to now, we have assumed that both  $\mathbb{G}_1$  and  $\mathbb{G}_2$  are undirected. In this section, we show through simulations that when  $\mathbb{G}_1$  and  $\mathbb{G}_2$  are directed, more undesirable system behaviors may arise. In [14], some necessary and sufficient conditions based on directed neighbor relationship graphs have been constructed for reaching consensus in multi-agent systems with second-order agent dynamics without quantization. However, those conditions are not applicable to the case with quantization. We use again an example to illustrate.

We take both  $\mathbb{G}_1$  and  $\mathbb{G}_2$  to be the directed ring shown on the bottom of Fig. 1, which is balanced and contains a directed spanning tree [23]. The other simulation conditions are set to be the same as in the simulation in Section 4. Although the logarithmic quantizers are used, neither the agents' positions  $r_i(t)$  nor their velocities  $v_i(t)$  can be synchronized, which keep oscillating as shown in Fig. 7(a) and (b) respectively. Note that the same system without quantization satisfies the conditions stipulated in Theorem 1 of [14] and thus can get synchronized. So conditions for synchronized motions with directed  $\mathbb{G}_1$  and  $\mathbb{G}_2$  need to be further investigated in the future.

## 7. Concluding remarks

We have shown the effects of different quantizers on the steady-state behavior of teams of mobile agents with second-order dynamics. We have studied the performances of the chosen logarithmic and uniform quantizers respectively. It has been emphasized that for coordinating agents with higher-order dynamics, the quantization effects of various quantizers are different and undesirable system behavior, e.g. oscillations, may happen even when the same system without quantization is stable.

We are working on looking into more different quantization schemes. We are also interested in understanding how different nonstandard solutions to nonsmooth systems can be used in the analysis of the quantization effects. More coordination strategies other than the consensus-type algorithms will be studied in the future to obtain more general conclusions about the quantization effects on coordination tasks in multi-agent systems in general.

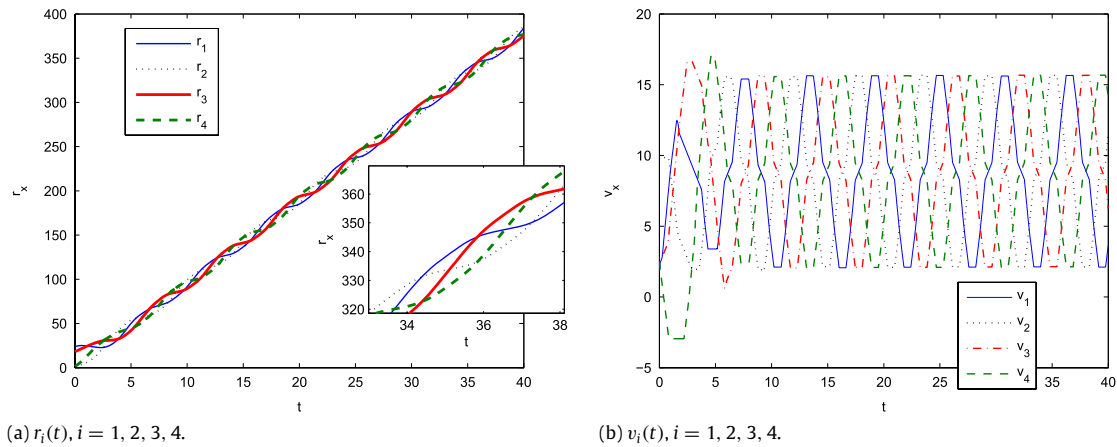


Fig. 7. Oscillating behavior when  $G_1$  and  $G_2$  are directed rings with the logarithmic quantizers.

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